1. Suppose we observe the random variable Y to estimate X. We know the following

   - $Y \triangleq X + N$
   - $X \sim N(\mu_x, \sigma_x^2)$
   - $N \sim N(0, \sigma_n^2)$
   - X and N are independent

   To calculate the MMSE estimator $\hat{x}$ in terms of random variable Y when $Y = y$, given by $g(y)$, we know from a derivation in our ECE 514 class that

   $$\hat{x} = g(y) = E[X|Y = y] = \int_{x=-\infty}^{\infty} x f_{X|Y}(x|y)dx \tag{1}$$

   where Bayes' theorem for probability density functions states

   $$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)f_X(x)}{f_Y(y)} \tag{2}$$

   The probability density of X is Gaussian, as given in the beginning of the problem. The probability density of Y is the sum of the two Gaussian's X and N. A helpful linear property of independent Gaussian's tells us that $Y \sim N(\mu_x + 0, \sigma_x^2 + \sigma_n^2)$.

   Finally, the probability density of $(Y|X = x)$ is simply $Y = x + N$, meaning that Y follows the distribution of N shifted by x, therefore $(Y|X = x) \sim N(x, \sigma_n^2)$

   All together,

   $$\frac{f_{Y|X}(y|x)f_X(x)}{f_Y(y)} = \frac{N(x, \sigma_n^2)N(\mu_x, \sigma_x^2)}{N(\mu_x, \sigma_x^2 + \sigma_n^2)} \tag{3}$$

   which equals

   $$\frac{\frac{1}{\sqrt{2\pi\sigma_n^2}} \exp \frac{-(y-x)^2}{2\sigma_n^2} \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp \frac{-(x-\mu_x)^2}{2\sigma_x^2}}{\frac{1}{\sqrt{2\pi(\sigma_x^2+\sigma_n^2)}} \exp \frac{-(y-\mu_x)^2}{2(\sigma_x^2+\sigma_n^2)}} \tag{4}$$

   Before calculating $\hat{x}$, we can try to simplify the quantity into another Gaussian, meaning equation (4) may be reproduced in the following form

   $$\frac{1}{\sqrt{2\pi v}} \exp \frac{-(x-m)^2}{2v} \tag{5}$$

   where m is the mean and v is the variance of the Gaussian distribution. Knowing this, we can focus separately on the exponential terms and the scalar multiples to reproduce it in the equation (5) format. The scalar component can be simplified down the following way

   $$\frac{\sqrt{2\pi(\sigma_x^2 + \sigma_n^2)}}{\sqrt{2\pi\sigma_n^2}\sqrt{2\pi\sigma_x^2}} \tag{6}$$

   $$\frac{\sqrt{2\pi}}{2\pi}\sqrt{\frac{\sigma_x^2 + \sigma_n^2}{\sigma_x^2\sigma_n^2}} \tag{7}$$

   $$\frac{1}{\sqrt{2\pi\frac{\sigma_x^2\sigma_n^2}{\sigma_x^2+\sigma_n^2}}} \tag{8}$$

So, for now, the quantity given above matches our format in equation (5), where

$$v_* = \frac{\sigma_x^2\sigma_n^2}{\sigma_x^2+\sigma_n^2} \tag{9}$$

however, we first must ensure we get the same variance in the exponential component before we make any conclusions. Exponents are added and subtracted when multiplying and dividing quantities respectively, so

$$f_{X|Y}(x|y) = \frac{\exp\left(\frac{-(y-x)^2}{2\sigma_n^2} + \frac{-(x-\mu_x)^2}{2\sigma_x^2} - \frac{-(y-\mu_x)^2}{2(\sigma_x^2+\sigma_n^2)}\right)}{\sqrt{2\pi\frac{\sigma_x^2\sigma_n^2}{\sigma_x^2+\sigma_n^2}}} \tag{10}$$

And, again, since we want the quantity above in the form given in equation (5), then we can safely play with the exponent to get it into the following form

$$\frac{-(x-m)^2}{2v} = \frac{-(x^2-2m+m^2)}{2v} \longleftrightarrow \left[\frac{-(y-x)^2}{2\sigma_n^2} + \frac{-(x-\mu_x)^2}{2\sigma_x^2} - \frac{-(y-\mu_x)^2}{2(\sigma_x^2+\sigma_n^2)}\right] \tag{11}$$

Knowing this, we can separate the right-hand side into terms of x. Time for some algebra

$$\frac{1}{2}\left[\frac{-(y^2-2yx+x^2)}{\sigma_n^2} - \frac{(x^2-2x\mu_x+\mu_x^2)}{\sigma_x^2} + \frac{(y-\mu_x)^2}{\sigma_x^2+\sigma_n^2}\right] \tag{12}$$

$$\frac{1}{2\sigma_x^2\sigma_n^2}\left[-(y^2-2yx+x^2)\sigma_x^2 - (x^2-2x\mu_x+\mu_x^2)\sigma_n^2 + \frac{(y-\mu_x)^2(\sigma_x^2\sigma_n^2)}{\sigma_x^2+\sigma_n^2}\right] \tag{13}$$

$$\frac{-1}{2\sigma_x^2\sigma_n^2}\left[x^2(\sigma_x^2+\sigma_n^2) + x(-2y\sigma_x^2-2\mu_x\sigma_n^2) + \left(y^2\sigma_x^2+\mu_x^2\sigma_n^2 - \frac{(y-\mu_x)^2(\sigma_x^2\sigma_n^2)}{\sigma_x^2+\sigma_n^2}\right)\right] \tag{14}$$

$$\frac{-(\sigma_x^2+\sigma_n^2)}{2\sigma_x^2\sigma_n^2}\left[x^2 - 2x\left(\frac{y\sigma_x^2+\mu_x\sigma_n^2}{\sigma_x^2+\sigma_n^2}\right) + \left(\frac{y^2\sigma_x^2+\mu_x^2\sigma_n^2}{\sigma_x^2+\sigma_n^2} - \frac{(y-\mu_x)^2(\sigma_x^2\sigma_n^2)}{(\sigma_x^2+\sigma_n^2)^2}\right)\right] \tag{15}$$

$$\frac{-\left[x^2 - 2x\left(\frac{y\sigma_x^2+\mu_x\sigma_n^2}{\sigma_x^2+\sigma_n^2}\right) + \left(\frac{y^2\sigma_x^2+\mu_x^2\sigma_n^2}{\sigma_x^2+\sigma_n^2} - \frac{(y-\mu_x)^2(\sigma_x^2\sigma_n^2)}{(\sigma_x^2+\sigma_n^2)^2}\right)\right]}{2\frac{\sigma_x^2\sigma_n^2}{\sigma_x^2+\sigma_n^2}} \tag{16}$$

Now only focusing on right side of numerator

$$\cdots + \left(\frac{(y^2\sigma_x^2+\mu_x^2\sigma_n^2)(\sigma_x^2+\sigma_n^2)}{(\sigma_x^2+\sigma_n^2)^2} - \frac{(y^2-2y\mu_x+\mu_x^2)(\sigma_x^2\sigma_n^2)}{(\sigma_x^2+\sigma_n^2)^2}\right) \tag{17}$$

$$\cdots + \left(\frac{y^2\sigma_x^4+y^2\sigma_x^2\sigma_n^2+\mu_x^2\sigma_x^2\sigma_n^2+\mu_x^2\sigma_n^4-y^2\sigma_x^2\sigma_n^2+2y\mu_x\sigma_x^2\sigma_n^2-\mu_x^2\sigma_x^2\sigma_n^2}{(\sigma_x^2+\sigma_n^2)^2}\right) \tag{18}$$

$$\cdots + \left(\frac{y^2\sigma_x^4+2y\mu_x\sigma_x^2\sigma_n^2+\mu_x^2\sigma_n^4}{(\sigma_x^2+\sigma_n^2)^2}\right) \tag{19}$$

$$\cdots + \left(\frac{y\sigma_x^2+\mu_x\sigma_n^2}{\sigma_x^2+\sigma_n^2}\right)^2 \tag{20}$$

So the numerator is

$$-\left[x^2 - 2x\left(\frac{y\sigma_x^2+\mu_x\sigma_n^2}{\sigma_x^2+\sigma_n^2}\right) + \left(\frac{y\sigma_x^2+\mu_x\sigma_n^2}{\sigma_x^2+\sigma_n^2}\right)^2\right] \tag{21}$$

which is in the form of

$$-(x-m)^2 \longleftrightarrow -\left(x - \frac{y\sigma_x^2 + \mu_x\sigma_n^2}{\sigma_x^2 + \sigma_n^2}\right)^2 \tag{22}$$

meaning, getting our exponential denominator back from equation (16) and our scalar component from equation (8)

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi\frac{\sigma_x^2\sigma_n^2}{\sigma_x^2+\sigma_n^2}}} \exp\left[\frac{-\left(x - \frac{y\sigma_x^2+\mu_x\sigma_n^2}{\sigma_x^2+\sigma_n^2}\right)^2}{2\frac{\sigma_x^2\sigma_n^2}{\sigma_x^2+\sigma_n^2}}\right] \tag{23}$$

which is in the form of a Gaussian outlined in equation (5), where

$$m = \frac{y\sigma_x^2 + \mu_x\sigma_n^2}{\sigma_x^2 + \sigma_n^2} \tag{24}$$

and

$$v = \frac{\sigma_x^2\sigma_n^2}{\sigma_x^2 + \sigma_n^2} \tag{25}$$

Since our expected value is in the form of a Gaussian distribution, we know the expected value, so we don't have to evaluate the integral, showing

$$\hat{x} = E[X|Y=y] = m = \frac{y\sigma_x^2 + \mu_x\sigma_n^2}{\sigma_x^2 + \sigma_n^2} \tag{26}$$

Now in order to estimate the random variable X, $\hat{x}$, using random variable Y, we can use

$$g(y) = \frac{y\sigma_x^2 + \mu_x\sigma_n^2}{\sigma_x^2 + \sigma_n^2} \tag{27}$$

2. Suppose $Z = X^2 + Y^2$ and $W = X$

   (a) Find the joint density of Z and W in terms of $f_{XY}(x,y)$
   The functions representing random variables Z and W are

$$g(x,y) = x^2 + y^2 \tag{28}$$

   and

$$h(x,y) = x \tag{29}$$

   respectively.
   Assuming that X and Y are real random variables, we can safely say that

$$f_{ZW}(z,w) = 0, \quad z < 0 \tag{30}$$

   And for $z \geq 0$, we can solve for x's and y's in terms of z's and w's

$$\begin{array}{rcl} x = w & \longrightarrow & x_1 = w, \quad y_1 = +\sqrt{z-w^2} \\ z = w^2 + y^2 & \longrightarrow & x_2 = w, \quad y_2 = -\sqrt{z-w^2} \end{array} \tag{31}$$

The Jacobian, defined as

$$J(x,y) = \begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix} \tag{32}$$

helps us in finding the PDF of the joint distribution. For values of x_1 and y_1,

$$|J(x_1,y_1)| = \left\| \begin{matrix} 2x_1 & 2y_1 \\ 1 & 0 \end{matrix} \right\| = |-2y_1| = 2\sqrt{z-w^2} \tag{33}$$

and similarly for x_2 and y_2,

$$|J(x_2,y_2)| = |-2y_2| = 2\sqrt{z-w^2} \tag{34}$$

Finally, our joint density can be represented in the following form

$$f_{ZW}(z,w) = \frac{f_{XY}(x_1,y_1)}{|J(x_1,y_1)|} + \frac{f_{XY}(x_2,y_2)}{|J(x_2,y_2)|} \tag{35}$$

$$f_{ZW}(z,w) = \frac{1}{2\sqrt{z-w^2}} \left[f_{XY}(w,\sqrt{z-w^2}) + f_{XY}(w,-\sqrt{z-w^2}) \right], \quad z \geq 0 \tag{36}$$

(b) Show that $f_W(w) = f_X(x)$ by evaluating

$$f_W(w) = \int_{z=-\infty}^{\infty} f_{ZW}(z,w)dz$$

We know that

$$W = X \tag{37}$$

So that

$$f_W(w) = \int_{z=-\infty}^{\infty} f_{ZX}(z,x)dz \tag{38}$$

And thus

$$f_W(w) = f_X(x) \tag{39}$$